STOCHASTIC CALCULUS WITH APPLICATIONS IN FINANCE (SEMINARS)

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This document have been typeset in LyX using article AMS class. Some visualization material downloaded from Wolfram and implemented using free Mathematica Player 7.01.

Some literature

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(2) van Handel, R., *Stochastic Calculus, Filtering, and Stochastic Control*, Lecture notes, 2007.
(3) Shreve, S., *Stochastic Calculus and Finance*, Lecture notes, 1997.
(4) Steele, J.M., *Stochastic Calculus and Financial Applications*, Springer 2000.
(5) Fries, C.P., *Mathematical Finance: Theory, Modeling and Implementation*, 2006. Simple and intuitive, provides numerous examples and intuitive explanations.
(6) Kuo, H.-H., *Introduction to Stochastic Integration*, Springer 2006. (Thank you Jet)
(7) *The Mathematics of Finance*, Lecture notes. (Thank you Shane)
(8) Varadhan,S.R.S., *Stochastic Processes*, Lecture notes, AMS 2007. (Thank you Mardi)
(9) Quastel,J., *Notes for Stochastic calculus for Mathematical Finance,* University of Toronto, `http://www.math.toronto.edu/quastel/fin.html`.
(10) Bass,R., *Lecture notes for Stochastic calculus, with applications to finance, PDE, and potential theory* , University of Connecticut, `http://www.math.uconn.edu/~bass/`.

Proposed Material (for future seminars)

(1) Probability modelling (σ-fields, information);
(2) Stochastic processes
(3) Brownian motion
(4) Non-differentiability, quadratic variation
(5) Conditional expectations, martingale theory, Markov property, stopping times
(6) Reflection principle, Dirichlet problem, recurrence and transience in $\mathbb{R}$
(7) Stochastic Calculus
    - Stochastic integrals

- SDE's
- Ito's lemma

(8) Ornstein-Uhlenbeck process, Levy process
(9) Local time, Feynman-Kac formula, Arcsin law
(10) More on SDE and their applications
- existence/uniqueness
- Bessel processes
- Backward and forward equations (BSDEs)

(11) Levy's theorem, time change and applications
(12) Cameron-Martin-Girsanov formula, Brownian bridge
- Hilbert space
- Radon–Nikodym derivative
- Risk neutral valuation

(13) Martingale representation theorem, applications
(14) Stochastic Optimal Control
- Hamilton–Jacobi–Bellman equation

(15) Diffusion limits of Markov chains
(16) Applications
- Hedging, arbitrage, option pricing
- Pricing portfolio of options;
- BSDE;
- Heston model

1. Preliminaries

1.1. **Basic model for probability.** We are interested in random variables, but to analyze them we actually have to start with modeling probabilities.

Random variables in $\mathbb{R}^n$: e.g. n random variables $X_1, ..., X_n$. You can just think of it as a random vector $X = (X_1, ..., X_n)$ with a distribution function $F(x_1, ..., x_n) = P(X_1 \leq x_1, ..., X_n \leq x_n)$.

Definition 1. A stochastic process is an indexed set of random variables $X_t, t \in T$, i.e. measurable maps from **probability space** $(\Omega, \mathscr{F}, P)$ to a **state space** $(E, \mathcal{E})\, T$.

Remark. Usually, $T = \mathbb{R}_+$(continuous time), but you could have $T = \mathbb{N}_+$(discrete time). Also, for our needs and purposes, $E = \mathbb{R}$ or $\mathbb{R}^d$ and $\mathcal{E} = \mathcal{B}\left(\mathbb{R}^d\right)$ is the Borel σ-field, i.e. the smallest σ-field containing all open sets.

Definition 2. A family of σ-fields, $(\mathscr{F}_t)_{t>0}$ is called the filtration.

The three ingredients of a probability model $(\Omega, \mathscr{F}, P)$

(1) What are all the things that could possibly happen?
- This is a set Ω;
- Every element $\omega \in \Omega$ symbolizes one possible fate of the model. For example, a coin flip $\Omega = \{heads, tails\}$ or a roll of two dice $\Omega = \{(1,1), (1,2), ..., (1,6), (2,1), ..., (6,6)\}$. Or position of a particle in a fluid $\Omega = \mathbb{R}^3$;
- Once we have specified Ω, any yes-no question is represented by the subset of Ω consisting of those $\omega \in \Omega$ for which the answer is yes. E.g. in a roll of a single dice, did we roll an even number, then subset is $\{2, 4, 6\}$;

(2) What sensible yes-no questions can we ask about these things?
- Specify yes-no questions to make sense and collect all sensible yes-no questions in a set $\mathscr{F}$, i.e. $\mathscr{F}$ is a set of subsets of Ω;
- For $s < t$, $(\mathscr{F}_s) \subset (\mathscr{F}_t)$ - i.e. you know more information at t then at s, information grows over time. Can have more then one filtration (e.g. insider trading: publicly available and privately available information);

(3) For any such question, what is the probability that the answer is yes?
- $P : \mathscr{F} \to [0; 1]$ s.t. $P(\Omega) = 1$

Randomness comes from choosing a *random* ω from Ω. In particular, choosing ω will determine the values of all random variables (i.e. random variables will be certain functions $X : \Omega \to \mathbb{R} \Rightarrow X(\omega)$.

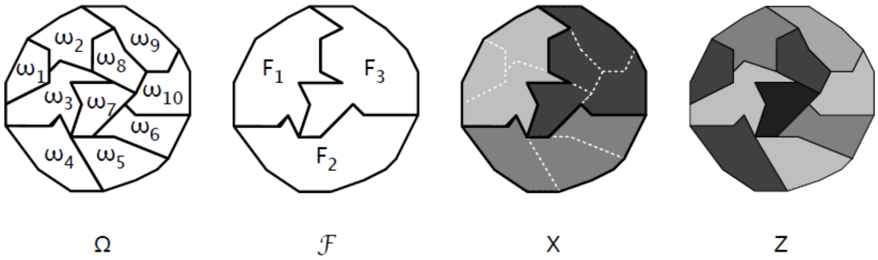


Figure 2.1.: *Illustration of Measurability: The random variables X and Z assign a grey value to each elementary event $\omega_1, \ldots, \omega_{10}$ as shown. The σ-algebra $\mathcal{F}$ is generated by the sets $F_1 = \{\omega_1, \omega_2, \omega_3\}$, $F_2 = \{\omega_4, \omega_5, \omega_6\}$, $F_3 = \{\omega_7, \ldots, \omega_{10}\}$. The random variable X is measurable with respect to $\mathcal{F}$, the random variable Z is not measurable with respect to $\mathcal{F}$.*

1.2. **Conditional Expectations.** TBA.

1.3. **Martingales.**

Definition 3. Let M_t be an integrable process adopted to $\mathscr{F}_t$ (e.g. M_t is $\mathscr{F}_t$-measurable $\forall t$ and $E[|M_t|] < \infty$). We say it is a martingale if $E[M_t | \mathscr{F}_s] = M_s \; \forall s \subset t$, or equivalently, $E[\triangle M | \mathscr{F}_s] = 0$.

Remark. On a related note: **Submartingale** is defined when $E[M_t | \mathscr{F}_s] \geq M_s$, or equivalently, $E[\triangle M | \mathscr{F}_s] \geq 0$. **Supermartingale** is defined when $E[M_t | \mathscr{F}_s] \leq M_s$, or equivalently, $E[\triangle M | \mathscr{F}_s] \leq 0$.

Example 4. Consider a gambler who wins \$1 when a coin comes up heads and loses \$1 when the coin comes up tails. Suppose now that the coin may be biased, so that it comes up heads with probability p.

- If p is equal to $1/2$, the gambler on average neither wins nor loses money, and the gambler's fortune over time is a martingale.
- If p is less than $1/2$, the gambler loses money on average, and the gambler's fortune over time is a supermartingale.
- If p is greater than $1/2$, the gambler wins money on average, and the gambler's fortune over time is a submartingale.

Example 5. A convex function of a martingale is a submartingale, by **Jensen's inequality**. For example, the square of the gambler's fortune in the fair coin game is a submartingale. Similarly, a concave function of a martingale is a supermartingale.

1.4. **Integral Calculus.** Let $f(.)$ denote a real valued non-negative map. $f(.)$ is called an elementary function if $f(.)$ takes on only a finite number of values $a_1, ..., a_n$. The Riemann integral partitions the domain of $f(.)$, the Lebesgue integral partitions the range of $f(.)$. For elementary functions both approaches give the same integral value. For general functions the corresponding integrals are defined as the limit of a sequence of approximating elementary functions, if these limits exist of course. Here, the two concepts are different: In the limit, all Riemann integrable functions are Lebesgue integrable and the two limits give the same value for the integral. However, there exist Lebesgue integrable functions for which the Riemann integral is not defined - i.e. its limit construction does not converge.

Example. One example of such a function is the indicator function of the rational numbers, also known as the Dirichlet function, $\mathtt{1}_{\mathbb{Q}}$.

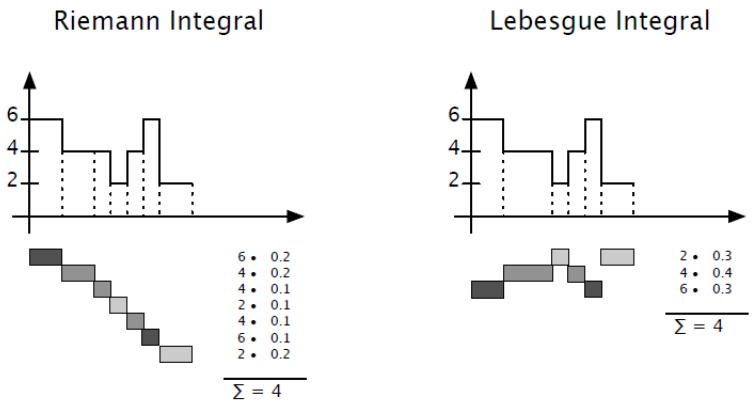


Figure 2.2.: *On the difference of Lebesgue and Riemann integral*

1.5. **Brownian Motion.**

Definition 6. Let $B : [0, \infty) \times \Omega \to \mathbb{R}^n$ denote a stochastic process with the following properties:

(a) $B_0 = 0$;
(b) For given $0 \le t_0 < t_1 < t_2 < ... < t_n$, increments $\triangle B_i = B_{t_i} - B_{t_{i-1}}$ are mutually independent random variables;
(c) $\triangle B_i \sim N(0, \triangle t_i)$ where $\triangle t_i = t_i - t_{i-1}$;
(d) The map $t \mapsto B_t$ is continuous almost surely (a.s.).

then B is called Brownian motion or a Wiener process (W).

- Could weaken (c) by saying that it is stationary: distribution of $\triangle B_i$ depends only on $\triangle t_i$ and not on actual values of t_i and t_{i-1}. For example, Levy processes.
- Note that (a), (b), (c) + stationary results in normality; (c) is just giving canonical form for mean and variance.
- We can abandon (d) and allow for price jumps.
- Existence of a process with such properties is non-trivial: if we replace (c) with log-normality there would be no such process. This is from the fact that the sum of two (independent) normal variables is normally distributed, but the sum of two lognormal variables is not lognormal.

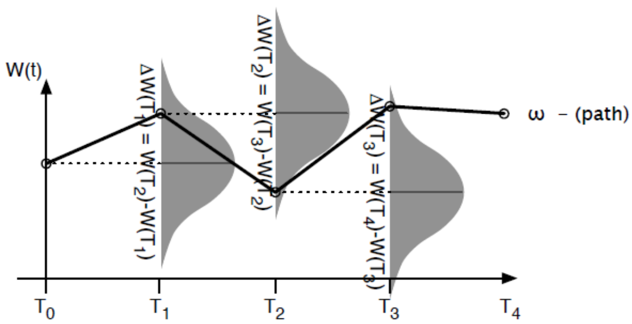


Figure 2.5.: *Time discretization of a Brownian motion: The transition $\Delta W(T_i) := W(T_{i+1}) - W(T_i)$ from time T_i to T_{i+1} is normal distributed. The mean of the transition is 0, i.e. under the condition that at time T_i the state $W(T_i) = x^*$ was attained, we have that the (conditional) expectation of $W(T_{i+1})$ is x^*:*

$$\mathrm{E}(W(T_{i+1}) \mid W(T_i) = x^*) = x^*.$$

Remark. $Cov(B_s, B_t) = s \wedge t$. Equivalent definition of Brownian Motion is a continuous Gaussian process with covariance $s \wedge t$.

Example 7. The Brownian motion may be viewed as the limit of a scaled random walk. Simple way to build a Brownian motion is to first build a random walk, $X_{n+1} = X_n \pm 1$ with probability $\frac{1}{2}$ each time. Connect the outcomes to make it a "time series" $\Rightarrow$ Z_t. Then rescale $Z_t^{(n)} = \frac{1}{\sqrt{n}} Z_{nt}$ and take limits as $n \to \infty$ to get a Brownian motion (see Donsker's theorem).

If we interpret the Brownian motion B for the movement of a particle, then B_t denotes the position of the particle at time t and $B_{t+\triangle t} - B_t$ the position change that occurs from t to $t + \triangle t$. To be precise, B_t models the probability distribution of the particle position (see Figure 2.5). Position changes $\sim N\left(0, \sqrt{\triangle t}\right)$. Mean 0 means that there is no directional preference in position change.

You can define a whole family of discrete stochastic processes. Suppose, following our example, a particle loses its energy over time: increments are still $\sim N$, but their standard deviation no longer $\sqrt{\triangle t_i}$. We can make it time dependant by scaling e.g. $e^{-t_i}\sqrt{\triangle t_i}$. Consider a process $X_{t_i} = \sum_{j=1}^{i} \sigma_{t_j} \triangle B_{t_j}$ where $\sigma_{t_j} = e^{-t_j}$.
Now, also suppose that a particle has a directional preference (e.g. gravity, or inflation), called a drift. We can add a constant mean, μ, or allow that μ to change over time. So, we add $\mu_{t_j} \triangle t_j$.
In addition, if you want, you can consider a starting point to be random as well.

$$X_{t_i} = X_0 + \sum_{j=1}^{i} \underbrace{\mu_{t_j} \triangle t_j + \sigma_{t_j} \triangle B_{t_j}}_{\sim N\left(\mu_{t_j} \triangle t_j, \sigma_{t_j} \sqrt{\triangle t_j}\right)}$$

$$X_{t_j} - X_{t_{j-1}} = \mu_{t_j} \triangle t_j + \sigma_{t_j} \triangle B_{t_j}$$

$$X_t - X_0 = \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s$$

1.6. **Geometric Brownian motion (GBM).** Assuming constant drift, μ, and volatility, σ, GBM is frequently presented in the following form:

$$dX_t = \mu X_t dt + \sigma X_t dB_t. \tag{1.1}$$

More generally, we could write

$$dX_t = a\left(X_t\right) dt + b\left(X_t\right) dB_t, \tag{1.2}$$

where GBM is just a special case when $a\left(X_t\right) = \mu X_t$ and $b\left(X_t\right) = \sigma X_t$.
Integrate both parts

$$X_t - X_0 = \int_0^t a\left(X_s\right) ds + \int_0^t b\left(X_s\right) dB_s$$

$$X_t = X_0 + \int_0^t a\left(X_s\right) ds + \int_0^t b\left(X_s\right) dB_s \tag{1.3}$$

where $\int_0^t E\left[b\left(X_s\right)^2\right] ds < \infty$ must be bounded to be deterministic.

For sufficiently smoothed functions $g\left(.\right)$ and $f\left(.\right)$ we can write

$$\int_0^t g\left(s\right) df\left(s\right) \approx \sum_{i=1}^{n} g\left(t_i\right) \triangle_i f$$

provided $g\left(.\right)$ is Borel measurable and $f\left(.\right)$ has finite variation

$$V_t = \lim_{n\to\infty} \sum \left|\triangle_i f\right| < \infty.$$

To make integral calculus work that's what we need, with finite variation being critical.

For Brownian motion

$$\sum_{i=1}^{n} |\triangle_i B| = \sqrt{\frac{t}{n}} \sum_{i=1}^{n} |z_i| \approx \sqrt{t}\sqrt{n} \to \infty$$

$$\triangle_i B \sim N(0, \triangle_i t) = \sqrt{\frac{t}{n}} z_i$$

Ito's idea was to stick a square in it (**quadratic variation** in Ito's formula):

$$\sum_{i=1}^{n} |\triangle_i B|^2 \to t$$

2. Stochastic Calculus

Definition 8. A solution to the SDE is a continuous adopted stochastic process X_t such that (1.3) holds.

Maps:

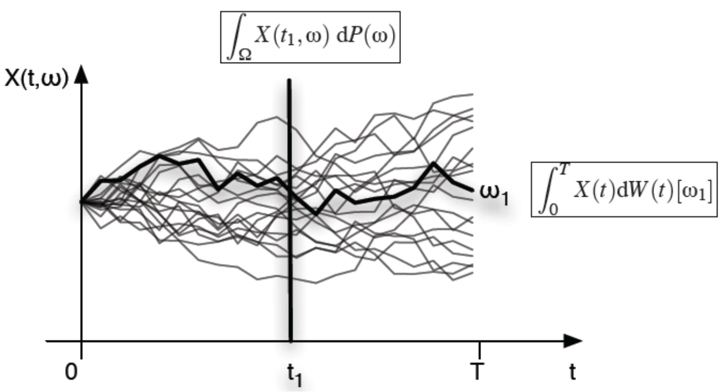
- $\int_{t_1}^{t_2} f(t)\,\mathrm{d}t$ – Lebesgue or Riemann integral. Integral of a real valued function with respect to t.

Random Variables:

- $\int_{\Omega} Z(\omega)\,\mathrm{d}P(\omega)$ – Lebesgue integral. Integral of a random variable Z with respect to a measure P (cf. expectation).

Stochastic Processes:

- $\int_{\Omega} X(t_1, \omega)\,\mathrm{d}P(\omega)$ – Lebesgue integral. Integral of a random variable $X(t_1)$ with respect to a measure P.
- $\int_{t_1}^{t_2} X(t)\,\mathrm{d}t$ – Lebesgue Integral or Riemann integral. The (pathwise) integral of the stochastic process X with respect to t.
- $\int_{t_1}^{t_2} X(t)\,\mathrm{d}W(t)$ – Itô integral. The (pathwise) integral of the stochastic process X with respect to a Brownian motion W.



2.1. **Ito's lemma.**

Theorem 9. *(Ito Lemma 1D). Let X denote an Ito process with*

$$dX_t = \mu dt + \sigma dB_t.$$

Let $f(t,x) \in C^2[0,\infty) \times \mathbb{R}$. Then we have that

$$Y_t \equiv f(t, X_t)$$

is an Ito process with

$$dY_t = f_t dt + f_x dX_t + \frac{1}{2} f_{xx} (dX_t)^2$$

where $(dX_t)^2 = dX_t \cdot dX_t$ is given by expansion with $dB_t \cdot dB_t = dt$, $dt \cdot dt = 0$, $dB_t \cdot dt = 0$ and $dt \cdot dB_t = 0$.

Another general representation of Ito's lemma:

Theorem 10. *(Ito Lemma). Let X denote an n-dimensional, m-factorial Ito process with*

$$dX_t = \mu dt + \sigma dB_t.$$

Let $f(t,x) \in C^2([0,\infty) \times \mathbb{R}^n; \mathbb{R}^d)$, $f = (f_1, ..., f_d)$. Then we have that

$$Y_t \equiv f(t, X_t)$$

is a d-dimensional, n-factorial Ito process with

$$dY_{k,t} = f_{k,t} dt + \sum_{i=1}^{n} f_{k,x_i} dX_{i,t} + \frac{1}{2} \sum_{i,j=1}^{n} f_{k,x_i x_j} (dX_{i,t})(dX_{j,t})$$

where $(dX_{i,t})(dX_{j,t})$ is given by expansion with $dB_{i,t} \cdot dB_{i,t} = dt$, $dB_{i,t} \cdot dB_{j,t} = \rho_{ij} dt$, $dt \cdot dt = 0$, $dB_{i,t} \cdot dt = 0$ and $dt \cdot dB_{i,t} = 0$.

TBA: add Product Rule and Quotient Rule for Ito.

Case 1. Let $X_t = f(B_t)$ where B_t is a Brownian motion and $f(x)$ is twice continuously differentiable, C^2, with bounded derivative, then

$$dX_t = f'(B_t) dB_t + \frac{1}{2} f''(B_t) dt$$
$$f(B_t) = f(B_0) + \int_0^t f'(B_s) dB_s + \frac{1}{2} \int_0^t f''(B_s) ds$$

Case 2. Let $X_t = f(t, B_t)$, then

$$dX_t = f_t(t, B_t) dt + f_x(t, B_t) dB_t + \frac{1}{2} f_{xx}(t, B_t) dt$$
$$f(t, B_t) = f(0, B_0) + \int_0^t f_s(s, B_s) ds + \int_0^t f_x(t, B_t) dB_t + \frac{1}{2} \int_0^t f_{xx}(s, B_s) ds$$

2.2. **Variations on Ito's lemma.**

Case 1. $f(x)$and $Y_t = f(B_t)$
$dY_t = f_x dB_t + \frac{1}{2} f_{xx} dt$

Case 2. $f(t,x)$and $Y_t = f(t, B_t)$
$dY_t = f_t dt + f_x dB_t + \frac{1}{2} f_{xx} dt$

Case 3. $Y_t = f(X_t)$, where X_t is a semimartingale
$dY_t = f_t dt + f_x dX_t + \frac{1}{2} f_{xx} dX_t \cdot dX_t$

Case 4. $Y_t = f(X_t^1, X_t^2, ..., X_t^n)$, where $X_t^1, X_t^2, ..., X_t^n$ are semimartingales.

3. Conditional Expectations

We start with a definiton of conditional probability. The probability of A given B is defined by

$$\mathbb{P}(A\,|B\,) \equiv \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

provided $\mathbb{P}(B) \neq 0$.
The conditional expectation of X given B is defined to be

$$\mathbb{E}[X\,|B\,] \equiv \frac{\mathbb{E}[X;B]}{\mathbb{P}(B)}$$

provided $\mathbb{P}(B) \neq 0$. Notation $\mathbb{E}[X;B]$ means $\mathbb{E}[X\mathbf{1}_B]$ where

$$\mathbf{1}_B(\omega) = \begin{cases} 1 & \omega \in B \\ 0 & otherwise \end{cases}$$

Another way of writing $\mathbb{E}[X;B]$ is

$$\mathbb{E}[X;B] = \sum_{\omega \in B} X(\omega)\,\mathbb{P}(\{\omega\})$$

Contrast with the following

$$\mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega)\,\mathbb{P}(\{\omega\})$$