Consumption-portfolio choice in continuous-time

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References

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Outline

I. Model
II. Dynamic problem
III. Static problem
IV. Optimal solution
V. Example

I. Model

- $(\Omega, \mathscr{F}, \mathbb{P})$: probability space, $W \in \mathbb{R}^d$: B-M, $\mathscr{F} \equiv \sigma(W)$.
- Financial markets:
  1. Riskiness stock (bound): $dB_t = r_t dt$, $r_t \equiv$ interest rate, r is progressively measurable (P.M), $\int_0^T r_t dt < \infty$ $\mathbb{P}$. a.s.
  2. risky stocks: $\underbrace{dS_t}_{d\times 1} = \underbrace{\mathbf{I}^S}_{d\times d}[\underbrace{\mu_t}_{d\times 1} dt + \underbrace{\sigma_t}_{d\times d} \underbrace{dW_t}_{d\times 1}] - \underbrace{D_t}_{d\times 1} dt$

$$\sigma = \begin{bmatrix} \sigma_{11} & \dots & \sigma_{1d} \\ \vdots & \dots & \vdots \\ \sigma_{d1} & \dots & \sigma_{dd} \end{bmatrix}$$

σ_{ij} : effect of stock i on B-M j, $\mathbf{I}^S$: diagonal, D : vector of dividends

I. Model

So, we can write

$$\underbrace{dS_t}_{\text{Gains in stocks}} + \underbrace{D_t dt}_{\text{Dividends}} = \underbrace{\mathbf{I}^S[\mu_t dt + \sigma_t dW_t]}_{\text{Itôt Process with drift } \mu \text{ and volatility } \sigma} \quad (1)$$

- (μ, σ) is P.M, $\int_0^T \mu_t dt < \infty$ $\mathbb{P}$. a.s, $\int_0^T \sigma_t dWt < \infty$ $\mathbb{P}$. a.s.
- $\sigma \in \mathscr{P}^2([W])$: $\int_0^T \sigma_t^2 dt < \infty$ $\mathbb{P}$. a.s $\Rightarrow \int_0^T \sigma_t dWt$ exists
- σ_t^{-1} exists for all $t \in [0, T]$: complete markets.

So, we have

$$dW_t = \sigma_t^{-1}[(\mathbf{I}^S)^{-1}(dS_t + D_t dt) - \mu_t dt]. \quad (2)$$

I. Model

Define $\theta_t = \sigma_t^{-1}(\mu_t - r_t \mathbb{1}_d)$: market price of risk or sharp ratio $\rightarrow$ expected excess returns per unity of risk

- $\int_0^T \theta'_v \theta_v dv < \infty$ $\mathbb{P}$. a.s, Novikov condition is satisfied: $\mathbb{E}(\eta_t) = 1$ for all $t \in [0,\ T]$ where $\eta_t = exp\left(-\frac{1}{2}\int_0^t \theta'_v \theta_v dv - \int_0^t \theta'_v dWv\right)$
- Investor: maximizes $\mathbb{E}[\int_0^T u(c_t, t)dt] = \mathscr{U}(c)$ $\rightarrow$ expected utility
- $u \uparrow, \frown$, obeys INADA, $u(c_t, t) \equiv$ stochastic process, $u(c_t, t) = exp\left(-\int_0^t \beta_v dv\right).v(c)$, β : process $\rightarrow$ stochastic discount factor
- Example: CRRA $\rightarrow$ $v(c) = \frac{1}{1-R}c^{1-R}$

I. Model

- Consumption policies: c is P.M, $\int_0^T c_t dt < \infty$ $\mathbb{P}$. a.s, $c \geq 0$ and $\mathscr{C} = \{c : c \text{ } is \text{ } P.M, \int_0^T c_t dt < \infty \mathbb{P}.a.s, \ c \geq 0\}$
- Portfolio policies: $\pi \in \mathbb{R}^d$, P.M, $\int_0^T [\pi_t'(\mu_t - r_t \mathbb{1}_d) + \pi_t' \sigma_t \sigma_t' \pi_t dt] < \infty$ $\mathbb{P}$. a.s, π : dollars invested in risky stock
- $X_t - \pi_t' \mathbb{1}_d$: dollars in riskiness stock, X_t : wealth

$$dX_t = \pi_t' (\mathbf{I}^S)^{-1} (dS_t + D_t dt) + (X_t - \pi_t' \mathbb{1}_d) r_t dt] - c_t dt$$
$$= (r_t X_t - c_t) dt + \pi_t' [(\mu_t - r_t \mathbb{1}_d) dt + \sigma_t dW_t], \ X_0 = x. \quad (3)$$

II. Dynamic problem

- Definition: A policy (c, π) is feasible if and only if $X_t \geq 0$ for all $t \in [0, T]$. Define $\mathscr{A}(x) = \{(c, \pi) : X_t \geq 0 \, \forall \, t \in [0, T]\}$
- $\mathscr{A}(x)$: space of feasible policies, no short sale is permitted but some π may be < 0.
- Definition: The dynamic problem of the investor is

$$\max_{(c,\pi)} \mathscr{U}(c) \text{ s.t. } (c, \pi) \in \mathscr{A}(x). \tag{4}$$

- Problem (4) could be solved by dynamic programming (Bellman) $\rightarrow$ complex
- there is a simple way to do that: statistic problem

III. Static problem

- Define $\xi_t = b_t \eta_t$: state price density or discount factor for cash-flows, $b_t = exp\left(-r_v dv\right)$: interest rate discount factor
- Define $\mathbb{Q} = \eta_T d\mathbb{P}$: risk neutral measure (Girsanov). Let $M_t = \xi_t S_t + \int_0^t \xi_v D_v dv$. Then, we can show that

$$\mathbb{E}_s^{\mathbb{Q}}(M_T) = M_s : \quad M \text{ is a martingale} \tag{5}$$

- Define $\mathscr{B}(x) = \{c \in \mathscr{C} : \mathbb{E}(\int_0^T \xi_v c_v dv) \leq x\} = \{c \in \mathscr{C} : \mathbb{E}^{\mathbb{Q}}(\int_0^T b_v c_v dv) \leq x\}$
- Definition: The static problem of the investor is

$$\max_c \mathscr{U}(c) \text{ s.t. } c \in \mathscr{B}(x). \tag{6}$$

III. Static problem

Theorem

(*i*) $(c, \pi) \in \mathscr{A}(x) \Rightarrow c \in \mathscr{B}(x)$

(*ii*) $c \in \mathscr{B}(x) \Rightarrow \exists \pi : (c, \pi) \in \mathscr{A}(x)$.

Note: dynamic problem $\Leftrightarrow$ static problem

IV. Static problem resolution

- The static problem is

$$\max_{c} \mathscr{U}(c) \text{ s.t. } c \in \mathscr{B}(x). \tag{7}$$

- At (t, w) the F.O.C. are

$$\underbrace{u'(c_t, t)}_{\text{marginal benefit in terms of utility}} = \underbrace{\lambda \xi_t}_{\text{marginal cost}} , \ (\lambda > 0), \tag{8}$$

$$\mathbb{E}\left(\int_0^T \xi_v c_v dv\right) = x , \tag{9}$$

V. Example

- Example: $u(c,t) = ln(c).\rho_t, \;\; \rho_t \leq 1$
- $u'(c_t,t) = c^{-1}\rho_t = \lambda\xi_t$ and $\hat{c}_t = (\frac{\lambda\xi_t}{\rho_t})^{-1}$. Hence $\mathbb{E}\left(\int_0^T \xi_v(\frac{\lambda\xi_v}{\rho_v})^{-1}dv\right) = x$, *i.e.* $\lambda^{-1} = \frac{x}{\mathbb{E}\left(\int_0^T \rho_v dv\right)}$. So,

$$\hat{c}_t = \frac{x}{\mathbb{E}\left(\int_0^T \rho_v dv\right)}.(\frac{\xi_v}{\rho_t})^{-1}, \tag{10}$$

$$\begin{aligned} X_t &= \mathbb{E}_t[\int_t^T \frac{\xi_t}{\xi_v}(\lambda\xi_v/\rho_v)^{-1}dv] \\ &= (\lambda\xi_t/\rho_t)^{-1}\mathbb{E}_t[\int_t^T \frac{\rho_v}{\rho_t}dv], \end{aligned} \tag{11}$$

V. Example

$$\hat{c}_t = \frac{x}{\mathbb{E}\left(\int_0^T \rho_v dv\right)}.\left(\frac{\xi_v}{\rho_t}\right)^{-1}, \tag{12}$$

$$(\lambda\xi_t/\rho_t)^{-1} = X_t.\frac{1}{\mathbb{E}_t[\int_t^T \frac{\rho_v}{\rho_t} dv]} = X_t.m_t\,, \tag{13}$$

$$\hat{c}_t = \underbrace{m_t}_{\text{MRC}}.X_t \tag{14}$$

We can also show that $\hat{\pi}_t = \underbrace{{m_t}^*}_{\text{MRI}}.X_t$