BEA400 Microeconomics

Lecture 11

Module 5: Choice Over Time with Uncertainty

Lecture 11: Stochastic Processes, Ito's Lemma and Stochastic Optimal Control

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Reading:
I haven't found a text that I'm happy with for this part of the course, so lecture notes should suffice. You could try:
Malliaris, A.G. and W.A Brock, *Stochastic Methods in Economics and Finance,* North-Holland, 1982. Chapters 2, 3 and 4.

Stochastic Processes

1. The Pure Weiner Process or Brownian Motion

A Pure or Basic Weiner Process or Brownian Motion refers to a continuous time stochastic process where a random variable at time *t*, $x(t) = x_t$ which evolves over a small interval of time dt, according to a stochastic differential equation,

$$dx = z\sqrt{dt}$$

where *z* is a standard normal with mean zero and standard deviation of 1.

Since *z* is normally distributed it follows that dx will be normally distributed with an expected value of zero and standard deviation of $\sqrt{dt}$. Thus *x* is changing randomly by dx since it depends on $z\sqrt{dt}$

$$dx \sim N\left(0, \sqrt{dt}\right)$$

That is dx is be normally distributed with $E(dx) = 0$ and $Var(dx) = dt$ (and $SD(dx) = \sqrt{dt}$)

The reason that dx is scaled with $\sqrt{dt}$ is that any other choice for the magnitude of dx would lead to a problem that is either meaningless or trivial when we consider what happens at the limit when $dt \to 0$.

Also if dx were not scaled in this way, the variance of the random walk would have a limiting value of 0 or ∞.

Now let us consider dx over two very small but consecutive time periods.

Consider two time periods t_0 and t_1, such that $T = t_0 + t_1$ with corresponding values of x, $x(t_0) = x_0$ and $x(t_1) = x_1$ and changes $dx(t_0) = dx_0$ and $dx(t_1) = dx_1$

With the change over both periods as $dx_T = dx_0 + dx_1$ then the expected value of *dx* over both periods is

$$E(dx_T) = E(dx_0) + E(dx_1) = 0 + 0 = 0$$

and variance

$$Var(dx_T) = Var(dx_0) + Var(dx_1) + 2Cov(dx_0, dx_1)$$

If we assume that the values of *z* are independent over time then dx in any period of time is independent of dx in all other periods and the variance is

$$\begin{aligned} Var(dx_T) &= Var(dx_0) + Var(dx_1) \\ &= dt_0 + dt_1 \\ &= dT \end{aligned}$$

More generally over a long time period *T,*

$$E(dx_T) = 0 \text{ and variance } Var(dx_T) = dT = \sum dt \text{ and } SD(dx) = \sqrt{\sum dt}$$

that is the variance of the Weiner process over the time period T, is equal to the sum of the changes in time periods.

This allows us to consider the value *x* over discrete changes in time $dt = 1$

Consider a starting point $s = 0$ then at time *T,* after *T* changes of $dt = 1$ then the variance of dx_T

$$Var(dx_T) = dT = \sum_{s=0}^{T} dt = \sum_{s=0}^{T} 1 = T$$

At the starting point $s = 0$, so with $x_s = x_0$ the change in *x* over *T*, dx_T is

$$dx_T = x_T - x_0$$

and

$$x_T = x_0 + dx_T$$

If the process starts at zero, $x_s = x_0 = 0$, then the value of *x* in an period *T > s > 0,* is given by

$$\begin{aligned} x_T &= x_0 + dx_T \\ &= dx_T \end{aligned}$$

Thus from

$dx \sim N\left(0, \sqrt{dt}\right)$

$E(dx) = 0$ and variance $Var(dx) = dt \;\; SD(dx) = \sqrt{dt}$

with the assumptions of $s = 0$, $x_s = x_0 = 0$ and $dt = 1$

we obtain that *x* in any future period *T* is

$x_T \sim N\left(0, \sqrt{T}\right)$

$E(x_T) = 0$ and variance $Var(x_T) = T \;\; SD(x_T) = \sqrt{T}$

and for any time period in between *s* and *T*, *t* $x_t \sim N\left(0, \sqrt{t}\right)$

2. Scaling the variance of the Basic Weiner Process

The Basic Weiner Process can be enhanced to scale the standard deviation of the random variable x_t by σ

$dx = \sigma z \sqrt{dt}$.

In which case $x_t \sim N\left(0, \sigma\sqrt{t}\right)$

Note that σ can be modelled to depend on time, *t*, and the value of x_t such that $\sigma = \sigma(t, x_t)$

3. Weiner Process or Brownian Motion with Drift

The Basic Weiner Process can also be enhanced by adding a drift or growth term, μ to the stochastic differential equation.

$$dx = \mu dt + \sigma z\sqrt{dt}$$

In which case $x_t \sim N\left(\mu t, \sigma\sqrt{t}\right)$

Note that as with σ, μ can be modelled to depend on time, *t*, and the value of x_t such that $\mu = \mu(t, x_t)$

4. Markovian Process

A Markovian process is one where the probability values of future values of *x* conditional on being at time *t*, only depend upon the current value of *x* and no other information.

$$P\left(x(t),t\right) = P\left(x(t),t \middle| x(t) = x_t\right)$$

While this might seem rather restrictive it can be modified to allow a fixed amount of past information. The General Weiner Process described below is an example of a Markovian process.

5. General Weiner Processes or Brownian Motions

General Weiner Process or Brownian Motion refers to a continuous time stochastic process where a random variable $x(t) = x_t$ evolves over time, *t*, according to some stochastic differential equation:

$$dx = \mu(t, x_t)dt + \sigma(t, x_t)z\sqrt{dt}$$

This also called an Ito Process.

Stochastic Integration

Stochastic Integration was developed by Ito(1944) who generalised the stochastic integral first introduced by Weiner (1923)

Consider for which a stochastic process $dx(t,x,z)$ with deterministic component and a random component which follows a Standardised Weiner process z.

$$dx = \mu(t,x_t)dt + \sigma(t,x_t)z\sqrt{dt}$$

Stochastic Integration transforming the above into an integral equation we get,

$$x_t = x(0) + \int_0^t \mu(s,x_s)ds + \int_0^t \sigma(s,x_s)z\sqrt{ds}$$

We have encountered integrals of the form $\int_0^t \mu(s,x_s)ds$ before but how do we cope with $\int_0^t \sigma(s,x_s)z\sqrt{dt}$ which does not exist? Ito's Lemma!

Ito's Lemma

Suppose x follows a Brownian Motion with

$$dx = \mu(t, x_t)dt + \sigma(t, x_t)z\sqrt{dt} \text{ and}$$
$$dx^2 = \sigma(t, x_t)^2 z^2 dt \text{ as } (dt)^2 = 0 \text{ and } dz \times dt = 0 \text{ and } (dz)^2 = dt$$

If the dynamics of $x(t)$ can be written by an Ito Process, then the dynamics of well-behaved function of $x(t)$ that describe its distribution, $y_t = F(t, x_t)$ will also be described by an Ito Process.

Ito's Lemma gives:

$$dy = F_x dx + F_t dt + \tfrac{1}{2}F_{xx}dx^2$$

$$dy = \left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right)dt + F_x\sigma dz$$

and expectation

$$E[dy/dt] = \left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right) + F_x\sigma E[dz]/dt$$
$$= \left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right) \qquad \text{as } E[dz] = 0$$

$$E[dy] = E\left[\left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right)dt + F_x\sigma dz\right]$$
$$= \left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right)dt$$
$$E\left[dy^2\right] = E\left[\left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right)^2 dt^2 + F_x\sigma\left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right)dzdt + F_x^2\sigma^2 dz^2\right]$$
$$= \left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right)^2 dt^2 + F_x^2\sigma^2 E\left[dz^2\right]$$

$$Var[dy] = F_x^2\sigma^2 E\left[dz^2\right] \qquad \text{as } E\left[dz^2\right] = 1$$
$$= F_x^2\sigma^2$$

Demonstration of Ito's Lemma

Suppose x follows a Brownian Motion with

$$dx = \mu_0 x dt + \sigma_0 x dz \text{ and}$$
$$dx^2 = \sigma_0{}^2 x^2 dz^2 = \sigma_0{}^2 x^2 dt \text{ as } (dt)^2 = 0, dz \times dt = 0 \text{ and } (dz)^2 = dt$$

Consider $y = \ln x(t)$

Ito's Lemma $dy = F_x dx + F_t dt + \frac{1}{2} F_{xx} dx^2$

$$dy = \frac{1}{x} dx + \frac{1}{2}\left(-\frac{1}{x^2}\right) dx^2$$

Note that $dx^2 = \sigma_0{}^2 x^2 dz^2 = \sigma_0{}^2 x^2 dt$,

$$dy = \frac{1}{x}(\mu_0 x dt + \sigma_0 x dz) - \frac{1}{2x^2}\sigma_0^2 x^2 dt$$
$$= \left(\mu_0 - \frac{1}{2}\sigma_0^2\right) dt + \sigma_0 dz$$

which can be written

$$y(t) = y(0) + \int_0^t \left(\mu_0 - \tfrac{1}{2}\sigma_0^2\right) dt + \int_0^t \sigma_0 dz$$

which gives

$$y(t) = y(0) + \left(\mu_0 - \tfrac{1}{2}\sigma_0^2\right) t + \sigma_0 z(t)$$

Now substituting back $y = \ln x(t)$ by using $e^y = x(t)$ and $x(0) = e^{y(0)}$ gives

$$x(t) = x(0)\exp\left[\left(\mu_0 - \tfrac{1}{2}\sigma_0^2\right) t + \sigma_0 z(t)\right]$$

Some Properties Ito's Lemma

$$\int_0^T (a_1\sigma_1 + a_2\sigma_1) dz = a_1 \int_0^T \sigma_1 dz + a_2 \int_0^T \sigma_2 dz$$

$$E\left[\int_0^T \sigma(t) dz(t)\right]^2 = \int_0^T E\left[\sigma(t)\right]^2 dt \quad \text{if } \int_0^T E\left[\sigma(s)\right]^2 ds < \infty$$

Derivation of Ito's Lemma (Not examinable)

A Taylor series expansion of $y(t) = F(t, x(t))$ around y_0 gives *dy*

The expected value of *dy* can be computed (noting that $E[dz] = 0$) as

$$E[y_t - y_0] = F'(x_0)E[(x_t - x_0)] + \tfrac{1}{2}F''(x_0)E\left[(x_t - x_0)^2\right] + \text{higher order terms}$$

since $E[(x_t - x_0)] = \mu t$ and $E\left[(x_t - x_0)^2\right] = V[x_t - x_0] + \mu^2 t^2 = \sigma^2 t + \mu^2 t^2$

$$E[y_t - y_0] = F'(x_0)\mu t + \tfrac{1}{2}F''(x_0)\left(\sigma^2 t + \mu^2 t^2\right) + \text{higher order terms}$$

with the variance of $y_t - y_0$ ignoring higher order terms and using $(dt)^2 = 0$, $(dz)^2 = dt$ and $dz \times dt = 0$

so by setting t =1 allows the change in *y* to be given by

$$dy = \left(F'(x_0)\mu + \frac{\sigma^2}{2}F''(x_0)\right)dt + F'(x_0)\sigma dz$$

or if F also depends on t then

$$dy = \left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right)dt + F_x\sigma dz$$

$$y(t) = \left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right)t + F_x\sigma z(t)$$

and finally dividing by dt and taking the expectation (noting $E[dz] = 0$) gives

$$\begin{aligned} E[dy]/dt &= \left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right) + F_x\sigma E[dz]/dt \\ &= \left(F_t + F_x\mu + \tfrac{1}{2}F_{xx}\sigma^2\right) \end{aligned}$$

Applications of Ito's Lemma

A Stochastic Rate of Inflation

Suppose the rate of inflation is given by a basic Brownian Motion $\frac{dP}{P} = \mu_0 dt + \sigma_0 dz$

We can easily find the mean and variance of such a series,

$$E\left[\frac{dP}{P}\right] = \mu_0 E[dt] + \sigma_0 E[dz]$$
$$E\left[\frac{dP}{P}\right] = \mu_0 dt$$

The average proportionate change in prices is expected to be μ_0 multiplied by the time period over which it is considered. This can result can be re-expressed as the

$$E\left[\frac{dP/dt}{P}\right] = \mu_0$$
$$E\left[\frac{d\dot{P}}{P}\right] = \mu_0$$
$$E[\dot{\pi}] = \mu_0$$

Thus the expected continuos rate of change in inflation is μ_0

$$Var\left[\frac{dP}{P}\right] = E\left[\left(\frac{dP}{P}\right)^2\right] - \left\{E\left[\frac{dP}{P}\right]\right\}^2$$
$$Var\left[\frac{dP}{P}\right] = \mu_0^2 E\left[dt^2\right] + \mu_0\sigma_0 E[dt\ dz] + \sigma_0^2 E\left[dz^2\right] - \left\{\mu_0 E[dt]\right\}^2$$
$$Var\left[\frac{dP}{P}\right] = \sigma_0^2 dt \qquad \text{as } \mathrm{E}[dz] = 0,\ \mathrm{E}[dt\ dz] = 0 \text{ and } dt^2 = dz$$

The variance in the proportionate change in prices is σ_0^2 multiplied by the time period over which it is considered.

$$Var\left[\frac{dP/P}{dt}\right] = \frac{\sigma_0^2}{dt}$$
$$Var[\dot{\pi}] = \frac{\sigma_0^2}{dt}$$
$$SD[\dot{\pi}] = \frac{\sigma_0}{dz} \qquad \text{as } dz = \sqrt{dt}$$

While the Brownian Motion can readily provide the mean and variance for the proportionate change in prices if we want to know something about the price level in any particular period we will need to solve the differential equation $dP = \mu_0 Pdt + \sigma_0 Pdz$. The presence of *P,* which depends on time, in the drift term is no problem as we have learnt to solve such differential equations previously, but its presence with the *dz* term requires the use Ito's Lemma in order to solve for *P*(*t*).

Let $y(t) = \ln P(t)$ then by Ito's Lemma $dy = F_x dx + F_t dt + \frac{1}{2} F_{xx} dx^2$

$$\begin{aligned} dy &= \frac{1}{P} dP + \tfrac{1}{2}\left(-\frac{1}{P^2}\right) dP^2 \\ &= \frac{1}{P}\mu_0 Pdt + \frac{1}{P}\sigma Pdz - \frac{1}{2P^2}\sigma^2 P^2 dt \\ &= \left(\mu_0 - \frac{1}{2^2}\sigma^2\right) dt + \sigma dz \end{aligned}$$

All terms are independent of time and so can easily be integrated to provide *y*(*t*)

$$y(t) = \left(\mu_0 - \frac{1}{2^2}\sigma^2\right) t + \sigma z$$

$$P(t) = e^{y(t)} = P(0)\exp\left[\left(\mu_0 - \tfrac{1}{2}\sigma_0^2\right) t + \sigma_0 z(t)\right]$$

Real Rate of Return with Stochastic Inflation

If we also have an asset returning a nominal return of Q period that is growing at r then

$dP = \mu Pdt + \sigma Pdz$ and $dQ = rQdt$

Then defining $q = \frac{Q}{P}$ the real return then Ito's lemma gives

$$dq = \frac{\partial q}{\partial t} dt + \frac{\partial q}{\partial P} dP + \frac{\partial q}{\partial Q} dQ + \frac{1}{2}\left(\frac{\partial^2 q}{\partial P^2} dP^2 + \frac{\partial^2 q}{\partial Q^2} dQ^2 + 2\frac{\partial^2 q}{\partial Q \partial P} dQdP\right)$$

Since $(dt)^2 = 0$, $(dz)^2 = dt$ and , $dQdP$ and dQ^2 equal zero.

Which allows

$$dq = -\frac{Q}{P^2}dP + \frac{1}{P}dQ + \frac{1}{2}\left(\frac{2Q}{P^3}\right)P^2\sigma^2 dt$$

$$= -\frac{Q}{P^2}\left(\mu P dt + \sigma P dz\right) + \frac{1}{P}rQdt + \left(\frac{Q}{P}\right)\sigma^2 dt \qquad \text{since } q = \frac{Q}{P}$$

$$\frac{dq}{q} = \left(r - \mu + \sigma^2\right)dt - \sigma dz$$

Black-Scholes Option Pricing

Suppose there are three assets. The first with mean rate of return μ_{p_1} and SD σ_{p_1}, the second with a mean rate of return μ_{p_2} and SD σ_{p_2} and the third a risk free asset that earns a rate of return r per period.

The nominal value of the portfolio is

$$v(t) = n_1(t)p_1(t) + n_2(t)p_2(t) + np_3(t)$$

$$dp_1(t) = \mu_{p_1}p_1(t)dt + \sigma_{p_1}p_1(t)dz \text{ and } dp_2(t) = \mu_{p_2}p_2(t)dt + \sigma_{p_2}p_2(t)dz$$

$$dv = n_1dp_1 + n_2dp_2 + dnp_3$$

$$= n_1p_1\left(\mu_{p_1}dt + \sigma_{p_1}dz\right) + n_2p_2\left(\mu_{p_2}dt + \sigma_{p_2}dz\right) + r\ np_3dt$$

$$\frac{dv}{v} = \frac{n_1p_1}{v}\left(\mu_{p_1}dt + \sigma_{p_1}dz\right) + \frac{n_2p_2}{v}\left(\mu_{p_2}dt + \sigma_{p_2}dz\right) + r\ \frac{np_3}{v}dt$$

Defining w_1, w_2 and w_3 as the shares the three assets from of the portfolio's value so that $w_1 = \frac{n_1p_1}{v}$, $w_2 = \frac{n_2p_2}{v}$ and $w_3 = \frac{np_3}{v} = 1 - w_1 - w_2$ so that the above equation can be written

$$\frac{dv}{v} = w_1\left(\mu_{p_1}dt + \sigma_{p_1}dz\right) + w_2\left(\mu_{p_2}dt + \sigma_{p_2}dz\right) + \left(1\text{-w}_1 - w_2\right)r\ dt$$

Defining w_1, w_2 for a riskless portfolio

$$\text{var}\left[\frac{dv}{v}\right] = w_1\sigma_{p_1}dz + w_2\sigma_{p_2}dz = 0$$

then $\quad w_1\sigma_{p_1}dz = -w_2\sigma_{p_2}dz \qquad \Rightarrow \quad \frac{w_1}{w_2} = -\frac{\sigma_{p_2}}{\sigma_{p_1}}$

$$\frac{dv}{v} = w_1\mu_{p_1}dt + w_2\mu_{p_2}dt + r\ \left(1\text{-w}_1 - w_2\right)dt$$

since $\frac{dv}{v} = rdt$ for the riskless portfolio

$$\left(w_1\left(\mu_{p_1} - r\right) + w_2\left(\mu_{p_2} - r\right)\right)dt = 0 \qquad \Rightarrow \frac{w_1}{w_2} = -\frac{\left(\mu_{p_2} - r\right)}{\left(\mu_{p_1} - r\right)}$$

Combining the two

$$\frac{w_1}{w_2} = -\frac{\sigma_{p_2}}{\sigma_{p_1}} = -\frac{\left(\mu_{p_2} - r\right)}{\left(\mu_{p_1} - r\right)} \quad \text{and so}$$

$$\frac{\left(\mu_{p_1} - r\right)}{\sigma_{p_1}} = \frac{\left(\mu_{p_2} - r\right)}{\sigma_{p_2}}$$

Thus the rate of return per unit of risk must be equal for the two risk assets.

If asset 2 is stock option whose price is determined by $p_2 = F\left(p_1, t\right)$ and $dp_1 = \mu_{p_1}dt + \sigma_{p_1}dz$

Using Ito's Lemma and that the rate of return per unit of risk must be equalised

$$dp_2 = F_t dt + F_{p_1}dp_1 + \tfrac{1}{2}F_{p_1p_1}dp_1^2$$
$$dp_2 = \left(F_t + \mu_{p_1}F_{p_1} + \tfrac{1}{2}F_{p_1p_1}\sigma_{p_1}^2 p_1^2\right)dt + F_{p_1}\sigma_{p_1}$$
$$dp_2 = F_t + rp_1F_{p_1} - rF + \tfrac{1}{2}F_{p_1p_1}p_1^2\sigma_{p_1}^2$$

If the option can only be exercised at terminal time T with exercise price p_T the boundary condition $F\left(0, s\right) = s, \;\; s = T - t$ and $F\left(p_1, T\right) = \max\left[0, p_1 - p_T\right]$

Then Black and Scholes (1973) and Merton (1973) demonstrated that

$$p_2\left(s, p_1, r, \sigma_{p_1}^2\right) = p_1\phi\left(d1\right) + \mathrm{e}^{-rs}\,\phi\left(d2\right)$$
$$d1 = \left[\ln\frac{p_1}{p_T} + \left(r + \frac{\sigma_{p_1}^2}{2}\right)s\right]\frac{1}{\sigma_{p_1}\sqrt{s}}$$
$$d2 = d1 - \sigma_{p_1}\sqrt{s}$$
$$\phi\left(y\right) = \frac{1}{2\pi^{\frac{1}{2}}}\int_{-\infty}^{y} e^{-s^2/2}ds$$

Note that $\phi\left(y\right)$ is the cumulative normal distribution

Stochastic Optimal Control Theory

Previously our equation of motion was written,

$$\dot{y} = f(u, y, t)$$

Now consider a stochastic differential equation where $\dot{y} = f(u, y, t)$ is the deterministic (non-random) component and $\sigma(t, y, u)dz$ is the stochastic component with $\sigma(t, y, u)$ being a deterministic function and dz being a Brownian motion increment.

The problem then becomes

$$V(y(t), T) = \underset{u}{\text{Max}}\ \text{E}\left[\int_0^T F(t, y(t), u(t))dt\right]$$

subject to

$$dy = f(t, y, u)dt + \sigma(t, y, u)dz$$

The optimal value of the maximised problem can be written as

$$\begin{aligned} V(y_0, t_0) &= \underset{u}{\text{Max}}\, E\left[\int_{t_0}^{t_0+\Delta t} F(t_0, y_0, u)dt + \int_{t_0+\Delta t}^{T} F(t, y, u)dt\right] \\ &= \underset{u}{\text{Max}}\, E\left[F(t, y, u(t))\Delta t + V(y_0 + \Delta y, t_0 + \Delta t)\right] \end{aligned}$$

$$V(y, t) \cong \underset{u}{\text{Max}}\, E\left\{F(t, y, u)\Delta t + \text{V}(\text{y}+\Delta y, t + \Delta\text{t})\right\}$$

Note that following our previous section on Ito’s Lemma the above stochastic differential equation can be rewritten as,

$$\begin{aligned} dV &= V_t + V_y dy + \tfrac{1}{2}V_{yy}dy^2 \\ &= V_t + V_y f(t, y, u)dt + \tfrac{1}{2}V_{yy}\sigma(t, y, u)^2 dt + V_y\sigma(t, y, u)dz \end{aligned}$$

Assuming that $V(y, t)$ is twice differentiable we expand the function on the right around (y, t) by Talyor Series expansion:

$$\text{V}(\text{y}+\Delta y, t + \Delta\text{t}) = \text{V}(\text{y}, t) + V_y\Delta y + \tfrac{1}{2}V_{yy}(\Delta y)^2 + V_t\Delta t + \tfrac{1}{2}V_{tt}(\Delta t)^2 + V_{yt}\Delta y\Delta t + h.o.t$$

Inserting $\Delta y = f(t,y,u)\Delta t + \sigma(t,y,u)\Delta z$, using $(\Delta t)^2 = 0$, $(\Delta z)^2 = \Delta t$ and $\Delta z \Delta t = 0$ and then simplifying gives.

$$V(y,t) \cong \underset{u}{\text{Max}} E\left\{F(t,y(t),u(t))\Delta t + \text{V}(\text{y},t) + \left[V_t + V_y f(t,y,u) + \tfrac{1}{2}V_{yy}\sigma^2(t,y,u)\right]\Delta t + V_y \sigma \Delta z\right\}$$

Now tale the expectation of the above, the only stochastic term is Δz and its expectation is $E(\Delta z) = 0$. Then subtract $V(y,t)$ from both sides and divide through by Δt and let $\Delta t \to 0$.

$$0 = \underset{u}{\text{Max}}\left\{F(t,y(t),u(t)) + V_t + V_y f(t,y,u) + \tfrac{1}{2}V_{yy}\sigma^2(t,y,u)\right\}$$

$$-V_t = \underset{u}{\text{Max}}\left[F(t,y(t),u(t)) + V_y f(t,y,u) + \tfrac{1}{2}V_{yy}\sigma^2(t,y,u)\right]$$

Hamilton-Jacobi-Bellman (HJB) Equation

This is the Hamilton-Jacobi-Bellman (HJB) equation of stochastic control theory

$$-V_t = \underset{u}{\text{Max}}\left[F(t,y(t),u(t)) + V_y f(t,y,u) + \tfrac{1}{2}V_{yy}\sigma^2(t,y,u)\right]$$

Note that the co-state variable λ is V_y so that $\frac{\partial V(y,t,T)}{\partial y(t)} = V_y = \lambda(t)$, differentiating with respect to the state variable gives $\frac{\partial^2 V(y,t,T)}{(\partial y(t))^2} = V_{yy} = \lambda_y = \frac{\partial \lambda(t)}{\partial y(t)}$ so that the HJB equation can be written $-V_t = \underset{u}{\text{Max}}\left[F(t,y(t),u(t)) + \lambda\ f(t,y,u) + \tfrac{1}{2}\lambda_y \sigma^2(t,y,u)\right]$.

If the transformed Hamiltonian function $\tilde{H} = \tilde{H}(u,y,\lambda,\lambda_y)$ is

$$\tilde{H} = F(t,y(t),u(t)) + \lambda f(t,y,u) + \tfrac{1}{2}\lambda_y \sigma^2(t,y,u)$$

Then $-V_t = \underset{u}{\text{Max}}\tilde{H}$

Assuming $\frac{\partial \tilde{H}}{\partial u} = 0$ can be solved for the optimal choice $u^* = u^*(y,\lambda,\lambda_y)$ then inserting into the HJB

$$-V_t = \tilde{H}^*(y,\lambda,\lambda_y).$$

since $u^* = u^*\left(y, \lambda, \lambda_y\right)$

$$dy = f\left(t, y, \lambda, \lambda_y\right)dt + \sigma\left(t, y, \lambda, \lambda_y\right)dz$$

Then since $\tilde{H}_\lambda^* = f\left(t, y, \lambda, \lambda_y\right)$

$$dy = \tilde{H}_\lambda{}^* dt + \sigma(\;)dz$$

It can also be shown by using the definition of $\lambda = V_y$, the stochastic equation of motion for y the state variable and Ito's lemma that

$$d\lambda = -\tilde{H}_y{}^* dt + \sigma(\;)\lambda_y dz$$

then optimal HJB conditions are:

$\frac{\partial \tilde{H}}{\partial u} = 0$	Equation for optimal choice *u.*
$d\lambda = -\tilde{H}_y{}^* dt + \sigma(\;)\lambda_y dz$	Equation of motion for the co-state variable.
$dy = \tilde{H}_\lambda{}^* dt + \sigma(\;)dz$	Equation of motion for the state variable *y*.
$\lambda\left(T, y(T)\right) = 0$	Endpoint restriction.

Consumption-Savings Decision with Risky Income

Suppose that each period *t* the consumer receives an income $y(t)$ per period *t* it grows at constant rate of $dy = \mu_Y$ but with random component $\sigma_Y ydz$ where dz is a standardised Weiner process.

$$dy = \mu_y ydt + \sigma_y ydz$$

We could include *y* as a state variable, with a corresponding co-state variable and apply the rules of Stochastic Optimal Control. However it will be much simpler for us to use Ito's Lemma on *dy* first (see stochastic rate of inflation example) and replacing *y* with:

$$y(t) = y(0)\exp\left[\left(\mu_y - \tfrac{1}{2}\sigma_y^2\right)t + \sigma_y z(t)\right]$$

so that the wealth evolution now explicitly includes the variance of income.

$$\begin{aligned} dw &= r\ w(t)dt + y(t)dt - c(t)dt \\ &= r\ w(t)dt + y(0)\exp\left[\left(\mu_y - \tfrac{1}{2}\sigma_y^2\right)t + \sigma_y z(t)\right]dt - c(t)dt \end{aligned}$$

$$E\left[\dot{w}\right] = r\ w(t) + y(0)\exp\left[\left(\mu_y - \tfrac{1}{2}\sigma_y^2\right)t\right] - c(t)$$

since $E\left[z(t)\right] = 0$

with terminal conditions $w(0) = w_0,\ \ \mathrm{w}(T) = 0$.

Proceeding in this manner we can simply use the non-stochastic version of the Hamiltonian, which for this problem is

$$H\left[t,w(t),c(t),\lambda\right] = e^{-\delta t}\ln c(t) + \lambda(t)\Big(rw(t) + y(0)\exp\left[\left(\mu_y - \tfrac{1}{2}\sigma_y^2\right)t + \sigma_y z(t)\right] - c(t)\Big)$$

with Hamiltonian Conditions

H1: $\dfrac{\partial H}{\partial c(t)} = e^{-\delta t}\dfrac{1}{c(t)} - \lambda(t) = 0 \qquad \Rightarrow \qquad c(t) = \dfrac{1}{\lambda(t)}e^{-\delta t}$

H2: $\dfrac{\partial H}{\partial \lambda} = \dot{w} = \dfrac{dw(t)}{dt} = r\ w(t) - c(t) \quad \Rightarrow$

$$w(t)e^{-rt} = w_0 + y_0\int e^{\left[\left(\mu_y - \frac{1}{2}\sigma_y^2\right) - r\right]t}dt - \int e^{-rt}c(t)dt$$

H3: $\dot{\lambda} = \frac{d\lambda(t)}{dt} = -\frac{\partial H}{\partial w(t)} = -r\lambda(t) \qquad \Rightarrow \qquad \lambda(t) = \lambda(0)e^{-rt}$

$$c_0 = \frac{\delta\left(w_0 + \frac{y_0}{r-\left(\mu_y - \frac{1}{2}\sigma_y^2\right)}\left(1-e^{\left[\left(\mu_y-\frac{1}{2}\sigma_y^2\right)-r\right]T}\right) - w(T)e^{-rT}\right)}{\left(1-e^{-\delta T}\right)}$$

to go with $c(t) = c_0 e^{(r-\delta)t}$ and

$w(t) = w_0 e^{rt} - \frac{c_0}{\delta} e^{rt}\left(1-e^{-\delta t}\right) + \frac{y_0}{r-\left(\mu_y - \frac{1}{2}\sigma_y^2\right)} e^{rt}\left(1-e^{\left[\left(\mu_y-\frac{1}{2}\sigma_y^2\right)-r\right]t}\right)$ to give the solution for the control variable, consumption and state variable wealth.

We can see that income growth adds to initial consumption in that it effectively reduces the rate of discounting on future income by its growth.

While income uncertainty reduces initial consumption in that it effectively increases the rate of discounting on present value of future income due to its uncertainty.

If we had proceeded without solving for *dy* first, we would have had to use the stochastic version of the Hamiltonian, which in this problem would be

$H\left[t, w(t), c(t), \lambda, \lambda_w\right] = e^{-\delta t}\ln c(t) + \lambda(t)\left(rw(t) + \mu_y - c(t)\right) + \frac{1}{2}\lambda_w y^2 \sigma_y{}^2$

and its first order conditions.

Consumption-Savings Decision with Optional Risky Asset

<u>Logarithmic Utility</u>

Ignoring income for the moment, assume that there is a risk asset with rate of return r_a and SD of σ_a then evolution of wealth becomes.

$$\begin{aligned} dw &= \left[w(t)\left((1-a)r \;+ ar_a\right) - c(t)\right]dt + aw\sigma_a dz \\ &= \left[rw(t) + a(r_a - r)w - c(t)\right]dt + aw\sigma_a dz \end{aligned}$$

The Maximum Value Function is

$$V\left(y(t),T\right) = \underset{c(t)}{Max} \int_0^T e^{-\delta t} \ln c(t) dt$$

so that $-V_t = \underset{u}{\text{Max}}\left[F\left(t,y(t),u(t)\right) + V_y f(t,y,u) + \tfrac{1}{2}V_{yy}\sigma^2(t,y,u)\right]$ the stochastic optimal control problem is,

$$-V_t = \underset{c(t),a(t)}{\text{Max}}\left\{e^{-\delta t}\ln c(t) + V_w\left[\left(r + a(r_a - r)\right)w(t) - c(t)\right] + \tfrac{1}{2}\left(aw\sigma_a\right)^2 V_{ww}\right\}$$

Or replacing $V_w = \lambda$ and defining the stochastic Hamiltonian as

$$\tilde{H}\left[t,w(t),c(t),a(t),\lambda,\lambda_w\right] = e^{-\delta t}\ln c(t) + \lambda\left[\left(r + a(t)(r_a - r)\right)w(t) - c(t)\right] + \tfrac{1}{2}\lambda_w\left(a(t)w(t)\sigma_a\right)^2$$

$$\delta V = \underset{c(t),a(t)}{\text{Max}}\left\{\tilde{H}\left[t,w,c,a,\lambda,\lambda_w\right]\right\}$$

Examine the Hamiltonian Conditions (removing time (t) from the variables for clarity).

H1A: $\dfrac{\partial \tilde{H}}{\partial a} = \lambda r_a w - \lambda r w + \lambda_w a w^2 \sigma_a^2 = 0 \qquad \Rightarrow \qquad a^* = -\dfrac{\lambda(r_a - r)}{\lambda_w w \sigma_a^2}$

H1: $\dfrac{\partial \tilde{H}}{\partial c} = e^{-\delta t}\dfrac{1}{c} - \lambda = 0 \qquad \Rightarrow \qquad c = \dfrac{1}{\lambda}e^{-\delta t}$

H1 implies $\dfrac{\partial \tilde{H}}{\partial c} = e^{-\delta t}\dfrac{1}{c} - \lambda = 0$

$$F_c = \lambda = V_w$$
$$\frac{\partial}{\partial w} F_c = \frac{\partial}{\partial w} V_w$$
$$\frac{\partial c}{\partial w} \frac{\partial}{\partial c} F_c = V_{ww}$$
$$\frac{\partial F_c}{\partial c} = V_{ww} \frac{\partial w}{\partial c}$$
$$-\frac{\partial F_c}{\partial c} \frac{c}{F_c} = -\frac{V_{ww}}{V_w} w \frac{\partial w / w}{\partial c / c}$$
$$-c \frac{F_{cc}}{F_c} = -\frac{V_{ww}}{V_w} w \frac{\partial w / w}{\partial c / c}$$

where $-\frac{V_{ww}}{V_w} w = -\frac{\lambda_w}{\lambda} w$ is the Arrow-Pratt Measure Relative Risk Aversion RRA of wealth

$\frac{\partial w / w}{\partial c / c}$ the elasticity of wealth for consumption

$-c \frac{F_{cc}}{F_c} = -\frac{\partial F_c}{\partial c} \frac{c}{F_c}$ the intertemporal elasticity of substitution

Thus the intertemporal elasticity of substitution is equal to the product of Arrow-Pratt Measure Relative Risk Aversion for wealth and the elasticity of wealth for consumption.

Or

Thus the Arrow-Pratt Measure Relative Risk Aversion is equal to the product of and the elasticity of consumption for wealth and the intertemporal elasticity of substitution.

Since $-\frac{V_{ww}}{V_w} w = -\frac{\lambda_w}{\lambda} w$ the Arrow-Pratt Measure Relative Risk Aversion the optimal proportion invested in the risky asset is

$$a^* = -\frac{\lambda(r_a - r)}{\lambda_w w \sigma_a^2} = \frac{1}{RRA(w)} \frac{(r_a - r)}{\sigma_a^2}$$

Thus the proportion invested in the risky asset is inversely related to the Relative Risk Aversion of wealth and positively related to assets return over the risk free asset and negatively to its variance. Thus if RRA is constant then *a** will also be constant over life.

$$\frac{\partial F_c}{\partial c} = V_{ww}\frac{\partial w}{\partial c}$$
$$F_{cc}/F_c = -V_{ww}/V_w$$

For $\ln c(t)$ the intertemporal elasticity of substitution is

$$-c\frac{F_{cc}}{F_c} = -\frac{\partial F_c}{\partial c}\frac{c}{F_c} = -c\times\frac{-c^{-2}}{c^{-1}} = 1$$

$1 = RRA(w)\dfrac{\partial w/w}{\partial c/c}$ since $\dfrac{\partial w}{\partial c} = -1$ and so $1 = -\dfrac{c}{w}RRA(w)$ thus the Arrow-Pratt Measure Relative Risk Aversion for logarithmic utility is $RRA(w) = -\dfrac{w}{c}$ and

Returning to the other Hamiltonian Conditions

H2: $\dfrac{\partial H}{\partial \lambda} = \dot{w} = \dfrac{dw(t)}{dt} = rw(t) + a(t)(r_a - r)w(t) - c(t)$

H3: $\dot{\lambda} = \dfrac{d\lambda(t)}{dt} = -\dfrac{\partial H}{\partial w(t)} = -\lambda(t)\big(r + a(t)(r_a - r)\big) - \lambda_w a(t)^2\sigma_a^2 w(t)$

Inserting H1A: $a^* = -\dfrac{\lambda(r_a - r)}{\lambda_w w\sigma_a^2}$ into H2 and H3

H2:
$$\begin{aligned}\dot{w} &= rw(t) - \frac{\lambda(t)(r_a - r)}{\lambda_w w(t)\sigma_a^2}(r_a - r)w(t) - c(t)\\ &= rw(t) - \frac{\lambda(t)(r_a - r)^2}{\lambda_w\sigma_a^2} - c(t)\end{aligned}$$

Since

$$RRA(w) = -\frac{V_{ww}}{v_w} w$$

$$-\frac{w}{c} = -\frac{\lambda_w}{\lambda} w$$

$$c = \frac{\lambda}{\lambda_w}$$

$$\dot{w} = rw(t) - c(t)\left(\frac{(r_a - r)^2}{\sigma_a^2} + 1\right)$$

Now H3:

$$\dot{\lambda} = -\lambda(t)\left(r + -\frac{\lambda(t)(r_a - r)}{\lambda_w w(t)\sigma_a^2}(r_a - r)\right) - \lambda_w \left(\frac{\lambda(t)(r_a - r)}{\lambda_w w(t)\sigma_a^2}\right)^2 \sigma_a^2 w(t)$$

H3: $$= -\lambda(t) r + \frac{\lambda(t)^2 (r_a - r)^2}{\lambda_w w(t)\sigma_a^2} - \frac{\lambda(t)^2}{\lambda_w w(t)} \frac{(r_a - r)^2}{\sigma_a^2}$$

$$\dot{\lambda} = -r\lambda(t)$$

Thus $\lambda(t) = \lambda(0)e^{-rt}$ and H1 is now $c(t) = \frac{1}{\lambda_0} e^{(r-\delta)t}$ so $c(0) = \frac{1}{\lambda(0)}$ thus

$c(t) = c(0)e^{(r-\delta)t}$

$$\dot{w} = rw(t) - c(0)e^{(r-\delta)t}\left(\frac{(r_a - r)^2}{\sigma_a^2} + 1\right)$$

Denote $\ell = \left(\frac{(r_a - r)^2}{\sigma_a^2} + 1\right)$

$$\dot{w} - rw(t) = -c_0 \ell e^{(r-\delta)t}$$

$$e^{-rt}\left(\dot{w} - rw(t)\right) = -c_0 \ell e^{-\delta t}$$

Using $\int e^{f(x)} dx = \frac{e^{f(x)}}{f'(x)} + A$ to integrate the RHS $\int_{s=0}^{t} e^{-\delta s} ds = -\frac{1}{\delta} e^{-\delta t} + A_1$ gives

$$\int e^{-rt}\left(\dot{w}-rw(t)\right)=-c_0\ell\int_{s=0}^{t}e^{-\delta s}ds$$

$$w(t)e^{-rt}=c_0\ell\frac{1}{\delta}e^{-\delta t}+A_1-A_2$$

$$w(t)e^{-rt}=A+c_0\ell\frac{1}{\delta}e^{-\delta t}$$

setting t=0

$$A=w(0)-c_0\ell\frac{1}{\delta}$$

$$w(t)e^{-rt}=w(0)-c_0\ell\frac{1}{\delta}\left(1-e^{-\delta t}\right)$$

setting $t = T$

$$w(T)e^{-rT}=w(0)-c_0\ell\frac{1}{\delta}\left(1-e^{-\delta T}\right)$$

$$c_0=\frac{\left(w_0-w(T)e^{-rT}\right)}{\ell\frac{1}{\delta}\left(1-e^{-\delta T}\right)}$$

Note $\frac{\partial\ell}{\partial r_a}>0,\frac{\partial\ell}{\partial\sigma_a^2}<0$ thus initial consumption $\frac{\partial c_0}{\partial r_a}<0,\frac{\partial c_0}{\partial\sigma_a}>0$

Consumption growth is the same $c(t)=c(0)e^{(r-\delta)t}$

While the PV of wealth $w(t)e^{-rt}=w(0)-c_0\ell\frac{1}{\delta}\left(1-e^{-\delta t}\right)$ will be higher for higher r_a and lower σ_a^2.

Since the intertemporal elasticity of substitution (Arrow-Pratt degree of relative risk aversion for consumption) is constant and equal to one. In $RRA(w)=-\frac{w}{c}$ there is no parameter to describe degree of degree of risk aversion for wealth with the logarithmic utility function other than the wealth and consumption.

Hyperbolic Absolute Risk Aversion (HARA) class of utility function

However if we choose a different functional form for utility the picture is different. If we consider a Hyperbolic Absolute Risk Aversion (HARA) class of utility function.

$$u\left(c(t)\right) = \frac{1}{\theta} c(t)^{\theta}$$

For this HARA utility function the intertemporal elasticity of substitution is

$$-\frac{\partial F_c}{\partial c}\frac{c}{F_c} = -c\frac{F_{cc}}{F_c} = -c \times \frac{(\theta-1)c^{\theta-2}}{c^{\theta-1}} = 1-\theta$$

$1-\theta = RRA(w)\frac{\partial w / w}{\partial c / c}$ since $\frac{\partial w}{\partial c} = -1$ and so $1-\theta = -\frac{c}{w}RRA(w)$ thus the Arrow-Pratt Measure Relative Risk Aversion for simple HARA utility is

$$RRA(w) = -(1-\theta)\frac{w}{c}$$

The corresponding stochastic Hamiltonian is

$$\tilde{H}\left[t, w(t), c(t), a(t), \lambda, \lambda_w\right] = e^{-\delta t}\frac{1}{\theta}c(t)^{\theta} + \lambda(t)\left[\left(r + a(t)(r_a - r)\right)w(t) - c(t)\right] + \tfrac{1}{2}\lambda_w\left(a(t)w(t)\sigma_a\right)^2$$

H1A: $\frac{\partial \tilde{H}}{\partial a} = \lambda r_a w - \lambda r w + \lambda_w a w^2 \sigma_a^2 = 0 \quad \Rightarrow \quad a^* = -\frac{\lambda(r_a - r)}{\lambda_w w \sigma_a^2}$

H1: $\frac{\partial \tilde{H}}{\partial c} = e^{-\delta t}c^{\theta-1} - \lambda = 0 \quad \Rightarrow \quad c = \left(\frac{1}{\lambda}e^{-\delta t}\right)^{\frac{1}{1-\theta}}$

$$\dot{\lambda} = -\lambda(t)\left(r + -\frac{\lambda(t)(r_a - r)}{\lambda_w w(t)\sigma_a^2}(r_a - r)\right) - \lambda_w\left(\frac{\lambda(t)(r_a - r)}{\lambda_w w(t)\sigma_a^2}\right)^2 \sigma_a^2 w(t)$$

H3: $$= -\lambda(t) r + \frac{\lambda(t)^2 (r_a - r)^2}{\lambda_w w(t)\sigma_a^2} - \frac{\lambda(t)^2}{\lambda_w w(t)}\frac{(r_a - r)^2}{\sigma_a^2}$$

$$\dot{\lambda} = -r\lambda(t)$$

Thus $\lambda(t) = \lambda(0)e^{-rt}$ and H1 is now $c(t) = \left(\frac{1}{\lambda(0)}e^{(r-\delta)t}\right)^{\frac{1}{1-\theta}}$ so $c(0) = \left(\frac{1}{\lambda(0)}\right)^{\frac{1}{1-\theta}}$ thus

$$c(t) = c(0)\left(e^{(r-\delta)t}\right)^{\frac{1}{1-\theta}}$$

Since

$$RRA(w) = -\frac{V_{ww}}{v_w}$$
$$-(1-\theta)\frac{w}{c} = -\frac{\lambda_w w}{\lambda}$$
$$\frac{c}{(1-\theta)} = \frac{\lambda}{\lambda_w}$$

Then substituting $\frac{\lambda}{\lambda_w} = \frac{c}{(1-\theta)}$ into $\dot{w}$ to give

$$\dot{w} = rw(t) - \frac{c(t)}{1-\theta}\frac{(r_a - r)^2}{\lambda_w \sigma_a^2} - c(t)$$

$$\dot{w} = rw(t) - c(t)\left(\frac{(r_a - r)^2}{\sigma_a^2}\frac{1}{(1-\theta)} + 1\right)$$

And substituting $c(t) = c(0)\left(e^{(r-\delta)t}\right)^{\frac{1}{1-\theta}}$

$$\dot{w} = rw(t) - c(0)\left(e^{(r-\delta)t}\right)^{\frac{1}{1-\theta}}\left(\frac{(r_a - r)^2}{\sigma_a^2}\frac{1}{(1-\theta)} + 1\right)$$

Denote $\ell = \left(\frac{(r_a - r)^2}{\sigma_a^2}\frac{1}{(1-\theta)} + 1\right)$

$$\dot{w} - rw(t) = -c_0 \ell e^{\frac{(r-\delta)}{1-\theta}t}$$

$$e^{-rt}\left(\dot{w} - rw(t)\right) = -c_0 \ell e^{\left(\frac{(r-\delta)}{1-\theta} - r\right)t}$$

Using $\int e^{f(x)}dx = \frac{e^{f(x)}}{f'(x)} + A$ to integrate the RHS $\int_{s=0}^{t} e^{\frac{(r-\delta)}{1-\theta}s}ds = \frac{1-\theta}{r-\delta}e^{\frac{(r-\delta)}{1-\theta}t} + A_1$ gives

$$\int e^{-rt}\left(\dot{w}-rw(t)\right)=-c_0\ell\int_{s=0}^{t}e^{\left(\frac{(r-\delta)}{1-\theta}-r\right)s}ds$$

$$w(t)e^{-rt}=-c_0\ell\frac{1-\theta}{r\theta-\delta}e^{\left(\frac{(r-\delta)}{1-\theta}-r\right)t}+A_1-A_2$$

$$w(t)e^{-rt}=A-c_0\ell\frac{1-\theta}{r\theta-\delta}e^{\left(\frac{(r-\delta)}{1-\theta}-r\right)t}$$

setting t=0

$$A=w(0)+c_0\ell\frac{1-\theta}{r\theta-\delta}$$

$$w(t)e^{-rt}=w(0)+c_0\ell\frac{1-\theta}{r\theta-\delta}\left(1-e^{\frac{r\theta-\delta}{1-\theta}t}\right)$$

Note that if $\Theta=0$ the logarithmic case

$$w(t)e^{-rt}=w(0)-c_0\ell\frac{1}{\delta}\left(1-e^{-\delta t}\right)$$

If $t=T$ then we can use the terminal condition for wealth to remove c_0 from the problem. If there was no terminal condition on wealth we would need to use the transversality conditions, (such as $\lambda_T=0$ if the terminal state is free) together with the equation of motion of the co-state variable to provide λ_0 and so c_0.

$$w(T)e^{-rT}=w(0)+c_0\ell\frac{1-\theta}{r\theta-\delta}\left(1-e^{\frac{r\theta-\delta}{1-\theta}T}\right)$$

$$c_0=\frac{\left(w_0-w(T)e^{-rT}\right)}{\ell\frac{1-\theta}{r\theta-\delta}\left(1-e^{\frac{r\theta-\delta}{1-\theta}T}\right)}$$